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by

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OSCILLOGRAPHS AND THEIR TESTS

BY A. E. KENNELLY, R. N. HUNTER AND A. A. PRIOR

ABSTRACT OF PAPER

A method and technique for the testing and calibration of oscillographs is described, using an auxiliary vibrator or "oscillographmeter" for the production of Lissajous optical figures, whereby the resonant frequency f_0 of the tested oscillograph may be readily ascertained. From this and one other test, which is preferably a comparative calibration at 60 ~ and at the resonant frequency, the bluntness of resonance B of the oscillograph is determined.

From the two essential constants f_0 and B of an oscillograph, its indications at any assigned frequency can be corrected for the inertia of its vibratory system. At high frequencies, the correction may be relatively large.

A number of oscillographs have been tested for their f_0 and B. The principal results obtained are reported in the paper.

ALTHOUGH oscillographs have come into very general electrical engineering use, and are invaluable in many laboratory investigations, no methods for testing them seem to have been published. It is important to develop methods for the testing of oscillographs, because all those oscillographs which employ mechanical vibratory systems behave differently to alternating currents of different frequencies. If calibrated at say 60 ~, their calibration at other frequencies, and especially at high harmonic frequencies, will be different. If an oscillographic waveform is analyzed into its Fourier components, the apparent value of each component requires to be corrected, both as to amplitude and as to phase. In some cases, these corrections may be very large; while in others they may be insignificant; but in all cases the oscillographer should know, and be able to determine, the magnitude of the correction to be applied to the terms of his Fourier analysis, if only for the purpose of assuring himself that these corrections are too small to be worth taking into account, for those purposes to which his oscillographic records are to be applied. Strictly speaking, no oscillogram is complete, as a technical record, without an appended index of correction. It has already been pointed out that these corrections are due*; but no technique has been published, so far as the authors are aware, for determining the range of frequency errors in any particular instrument. It is the object of this paper to supply a technique for the testing of oscillographs, as the outcome of laboratory researches conducted to that end for the last two years.

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*Bibliography 7, 12.

Before discussing the technique of oscillograph testing, it is desirable to consider certain preliminary and underlying principles relating to the operation of the mechanically vibrating oscillograph. These principles are by no means new†; but the forms in which they are here presented are believed to be distinctive, and relatively easy for the student to apprehend. They depend upon a close analogy between the mechanics of a simple vibratory system and the electromagnetics of a simple alternating-current system.

Vector Current Impedance of an L. R. S. Branch. Fig. 1 represents a simple branch circuit $A\,B$, between a pair of alternating-current mains $m\,m'$, which are maintained at a constant r.m.s. voltage, by means of voltmeter V. The branch contains a condenser of capacitance C farads, a resistance of R ohms and an inductance of L henrys. Any resistance associated with the inductance is supposed to be displaced into, and segregated with, the resistance R. The values of C, L and R, are assumed to remain fixed and unchanged.

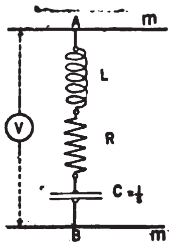

Fig. 1–Single *LRS* Branch Circuit

The a-c. generator connected to and supplying the mains $m\,m'$, is supposed to be able to supply, at will, any desired frequency, from the lowest to the highest, without altering the r.m.s. e.m.f. E volts. This e.m.f., while being varied in frequency from nearly zero to a very high value, remains simply sinusoidal throughout. In other words, the impressed e.m.f. is supposed to be pure, and devoid of harmonics. Then, at any impressed frequency f cycles per second, and impressed angular velocity $\omega = 2\,\pi\,f$ radians per second, the impedance of the branch circuit will be

$$Z = R + j\left(L\,\omega - \frac{1}{C\,\omega}\right) \qquad \text{ohms } \angle \qquad (1)$$

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†Bibliography 2, 4, 5, 7a, 8, 12.

or, if $S = 1/C$, is the elastance of the condenser in darafs, this becomes

$$Z = R + j\,(L\,\omega - S/\omega) \qquad \text{ohms} \angle \qquad (2)$$

the graph of this impedance, as is well known, is the straight line $A\,B\,C\,D\,E\,F$, Fig. 2, in the impedance plane, parallel to the Y axis, and distant R ohms therefrom. Commencing with frequency zero, the impedance $j\,L\,\omega$ of the reactor is indefinitely small, but the impedance $-\,j\,S/\omega$ of the condenser is infinite; so that the impedance Z then has the size of infinity, in the direction of $B\ A$ produced, and with a slope α of $-$ 90 deg. As the impressed angular velocity is increased, both the size and the slope of Z diminish; until, at a certain frequency, which may be denoted by ω_1, the size of the reactance $(L\,\omega - S/\omega)$ is negatively just equal to the size of the resistance R. The impedance will then be $O\,B$ Fig. 2, or

$$Z_1 = R - j\,R = R\,\sqrt{2}\;\overline{\diagdown}\;45 \qquad \text{ohms} \angle \qquad (3)$$

the slope α being then $-$ 45.

Increasing the angular velocity up to the resonant value, which may be denoted by ω_0, the reactance $L\,\omega - S/\omega = 0$, and the impedance becomes the real value $O\,D = R$ ohms, of slope 0 deg.

Increasing the angular velocity up to a certain value ω_2, when the reactance has a size that is positive and just equal to R, the impedance reaches the vector value $O\,E$, and is:

$$Z_2 = R + j\,R = R\,\sqrt{2} \angle 45. \qquad \text{ohms} \angle \qquad (4)$$

The values of angular velocity ω_1 and ω_2, may be called the lower and upper *quadrantal angular velocities*, respectively. Similarly, Z_1 and Z_2, corresponding thereto, are the *quadrantal impedances*.

Finally, if the impressed angular velocity is increased towards infinity, the reactance of the condenser C becomes indefinitely small; but that of the inductance L indefinitely great; so that the impedance of the $C\,L\,R$ branch approaches infinity in size, and + 90 deg. in slope, and lies in the vector direction from O, towards $E\,F$ indefinitely produced.

The impedance diagram of Fig. 2 has been prepared for the case in which $R = 100$ ohms, $C = 0.5 \times 10^{-6}$ farad and $L = 5 \times 10^{-3}$ henry. Each unit on the scale of the diagram then represents 100 ohms. The resonant angular velocity ω_0 is thus 20,000 radians per second, corresponding to a frequency of $f_0 = 3183 \sim$. If we denote by u, the ratio of the impressed to resonant frequency, or

$$u = \omega / \omega_0 = f / f_0 \qquad \text{numeric} \qquad (5)$$

the small circles along the impedance locus, Fig. 2, are marked with their corresponding values of u, for this particular case, from $u = 0.2$ or $\omega = 4000$, to $u = 5$ or $\omega = 100{,}000$.

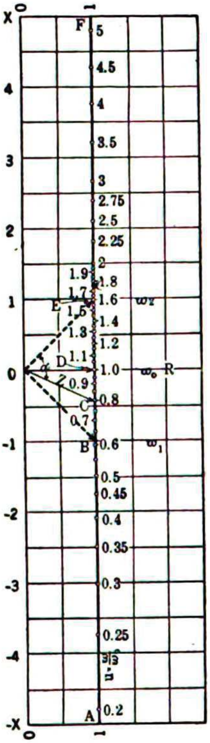

FIG. 2—RECTILINEAR GRAPH OF IMPEDANCE TO CURRENT FOR CASES OF $\left\{ \begin{array}{lll} R = 10^3. & L = 5 \times 10^{-2}, & S = 2 \times 10^6 \\ r = 10^{-2}, & m = 5 \times 10^{-7}, & s = 2 \times 10^2 \end{array} \right\}$ $w_0 = 20{,}000$ $\Delta = 10{,}000$ $B = 0.5$

IN THE ELECTRIC CASE, UNIT LENGTH = 10^2 OHMS.

IN THE MECHANIC CASE, UNIT LENGTH = 10^{-2} MECHANIC ABSOHMS, OR DYNES-PERP. CM. PER RADIAN PER SEC.

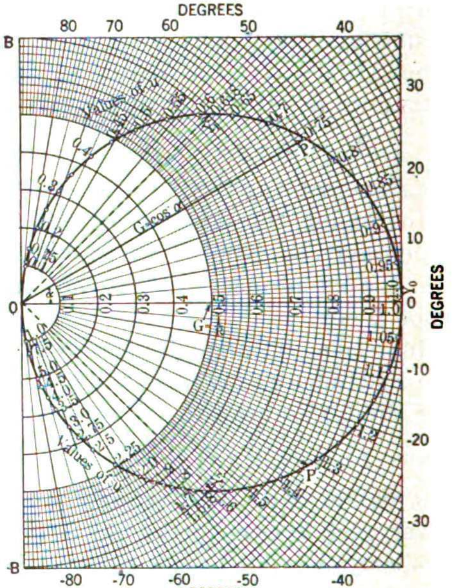

FIG. 3—CIRCULAR GRAPH OF ADMITTANCE TO CURRENT OR TO VELOCITY FOR CASES OF $\left\{ \begin{array}{lll} R = 10^2, & L = 5 \times 10^{-3}, & S = 2 \times 10^6 \\ r = 10^{-2}, & m = 5 \times 10^{-7}, & s = 2 \times 10^2 \end{array} \right\}$ $w_0 = 20{,}000$ $\Delta = 10{,}000$ $B = 0.5$

IN THE ELECTRIC CASE, UNIT LENGTH REPRESENTS 10^{-2} MHO.

IN THE MECHANIC CASE, UNIT LENGTH REPRESENTS 10^2 MECHANIC ABMHO OR RADIANS PER SEC. PER UNIT V.M.T.

The lower quadrantal value of impedance $O\,B$, occurs at $u = 0.618$, or $\omega_1 = 12{,}361$ radians per second. The upper value $O E$, occurs at $u = 1.618$, or $\omega_2 = 32{,}361$. The range between these two quadrantal values of impressed angular velocity may be described, for convenience as the quadrantal range.

As will be subsequently discussed, the quadrantal range of u in each case corresponds to the sharpness of resonance. In this case, the quadrantal range of u is from 0.618 to 1.618, or amounts to 1.0.

If we form the graph of the admittance Y, of the branch $L\,R\,S$, by forming the successive planevector reciprocals of the impedance Z, we obtain the circle $O\,Y_1\,Y_0\,Y_2$, Fig. 3, around which advance is made clockwise, as the impressed frequency is increased. $O\,Y_1$ and $O\,Y_2$, are the quadrantal admittances at the lower and upper quadrantal angular velocities ω_1 and ω_2 respectively. Any point in the Z graph, Fig. 2, characterized by the slope $-\alpha$ deg., of the planevector $O\,P$, corresponds to a point P, in the Y graph Fig. 3, having for its planevector $O\,P$ the slope $+\alpha$ deg, and the size $G \cos \alpha$, where $G\ (= 1/R)$, is the conductance of R in mhos.

It is easy to show that the two quadrantal angular velocities ω_1 and ω_2, whose vector admittance slopes are 45 deg. plus and minus respectively, have, as their geometrical mean, the resonant angular velocity, or

$$\sqrt{\omega_1\,\omega_2} = \omega_0 \qquad \frac{\text{radians}}{\text{sec.}} \qquad (6)$$

Moreover, this relation is not confined to the quadrantal pair of angular velocities. Any pair ω_1' and ω_2', characterizing the vectors $O\,P$ and $O\,P'$ in Fig. 3, whose slopes are respectively $+\alpha$ deg. and $-\alpha$ deg., are subject to the relation expressed in (6), or have ω_0 as their geometric mean.

Vector Maximum Cyclic Current in a C L R Branch. It will be evident from a consideration of the vector admittance graph Fig. 3, that if a maximum cyclic constant voltage E is impressed, at standard phase, on the $C\,L\,R$ branch, at varied frequency, the vector graph of maximum cyclic current in the branch will be a circle, similar to that of Fig. 3, since

$$I = E\,Y \qquad \text{max. cyc. amperes} \angle \qquad (7)$$

where I is the maximum cyclic vector current. In fact, Fig. 3 may then be interpreted as a vector current graph, if the scale of the diagram be properly chosen. Consequently, as the frequency is increased from zero to infinity, the maximum cyclic current changes from 0 ∠ 90 deg. to 0 ⦣ 90 deg. amperes, through the resonant value $E\,G$ ∠ 0 deg.

Corresponding Mechanical Vibrating System. Fig. 4 represents the elements of a simple bifilar, oscillographic vibrator, in which two similar parallel fine wires or strips, $A\,B$ and $C\,D$, carry the same alternating current in a uniform powerful magnetic field. The direction of the field lies substantially in the plane $A\,B\,D\,C$. At the center of the rectangle $A\,B\,D\,C$, the small mirror m is fastened symmetrically. The two wires are supposed to be equally stressed in tension. The vibrator is immersed in a damping fluid, such as air, oil, or glycerine.

When a maximum cyclic current I absamperes passes through the vibrator, it produces a maximum cyclic vibromotive torque on the mirror

$$F = A\,I \qquad \text{max. cyc. dyne-perp.-cm.} \ \angle \qquad (8)$$

This is a planevector alternating torque, proportional to I, and measurable in dynes acting perpendicularly to a radius arm of 1 cm., supposed to be erected perpendicularly to the plane of the mirror. The coefficient A is a real numeric, depending on the strength of the magnetic field, the length and the tension of the vibrator wires, etc. The phase of the torque will be the same as the phase of the current, if the wires are of non-magnetic material. The angular displacements produced in the vibrator mirror are assumed to be small; *i. e.*, of the order of a few degrees only.

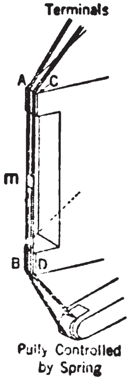

FIG. 4—OSCILLOGRAPH VIBRATOR

The alternating vibromotive torque F (abbreviated v.m.t.), acting on the vibrator of Fig. 4, corresponds to the alternating e.m.f. E, acting on the $C\,L\,R$ branch of Fig. 1. F tends to produce an alternating angular velocity of the mirror, just as E tends to produce an alternating current in the $C\,L\,R$ branch. Just as E is opposed by an electric impedance Z, expressed as in (2), so F, is opposed by a mechanical impedance z, expressed by a similar equation

$$z = r + j\,(m\,\omega - s/\omega) \qquad \frac{\text{dynes perp. cm.}}{\text{radians per sec.}} \ \angle \qquad (9)$$

Here r is the frictional resistance of the vibrator $A\,B\,C\,D$, Fig. 4, to alternating angular velocity, which causes energy to be dissipated as heat in the damping field. The quantity $(m\,\omega - s/\omega)$ is that mechanic resistance of the vibrator to alternating angular velocity, which causes energy to be shifted cyclically from the resilient wires to the inertia of the vibrating system, and which, therefore, occupies mechanical power in cyclic storage and release, but without dissipation. The quantity m corresponds to the inductance L of the branch circuit, and is the moment of inertia of the vibrating system, expressible in gm – cm^2, or what is equivalent thereto, in $\frac{\text{dynes-perp. cm.}}{(\text{radians per sec.})^2}$. The quantity s corresponds to the elastance S of the branch circuit, and is the elastic stiffness, or resistance to angular displacement, of the vibrator, expressible in dynes-perp.-cm. per radian. The complex quantity Z, is thus a mechanic impedance, analogous to electric resistance, and capable of being expressed in mechanic absohms $\angle$.

Vector Angular-Velocity Impedance. In view of the corresponding relations between an electric branch circuit, and a mechanic vibrator, the impedance graph of Fig. 2 may be regarded as representing the graph of impedance to angular velocity of the vibrator mirror in Fig. 4. At a very low impressed angular velocity, this impedance z commences at ∞ ⦢ 90 deg. mechanic absohms. At a very high impressed ω, z ends at $\infty \angle 90$ deg. mechanic absohms. At resonant angular velocity, ω_0, z passes through the value $r \angle 0$ deg.

Vector Angular-Velocity Admittance. In view of the same electric-mechanic analogy, the circular vector admittance graph to alternating current, Fig. 3, may also be regarded as a vector admittance graph to alternating torque. As the angular velocity impressed on the vibrator increases from zero to infinity, the admittance $y = 1/z$, varies from $0 \angle 90$ deg., through $g \angle 0$ deg., to 0 ⦢ 90 deg. mechanic abmhos, where $g = 1/r$, is the "mechanic conductance" of the system, expressible also in $\frac{\text{radians per sec.}}{(\text{dynes perp. cm.})}$.

The circular admittance graph of Fig. 3, corresponds not only to Fig. 2, by inversion; but also to either of the following cases: Electrically, $R = 100$ ohms, $G = 10^{-2}$ mho, $L = 5 \times 10^{-3}$ henry, $S = 2 \times 10^{6}$ darafs.

Mechanically, $r = 10^{-2} \dfrac{\text{dynes perp. cm.}}{\text{radian/sec.}}$,

$$g = 100 \frac{\text{radian/sec.}}{\text{dyne perp. cm.}},\ m = 5 \times 10^{-7}\ \text{gm.} - \text{cm.}^2,$$

$$s = 2 \times 10^2 \ \frac{\text{dyne perp. cm.}}{\text{radian}}$$

In either case, the resonant angular velocity $\omega_0 = 20{,}000$ radians/sec.

In the electric case, unit size on the graph corresponds to 10^{-2} mho.

In the mechanic case, unit size corresponds to

$$10^2 \ \frac{\text{radian/sec.}}{\text{dyne perp. cm.}} \cdot$$

The resonant range in u is from $u_1 = 0.618$, through 1.0 to $u_2 = 1.618$.

The size of the resonant admittance is G mhos electrically; or $g \dfrac{\text{radians/sec.}}{\text{dyne perp. cm.}}$ mechanically; or what may be described as "mechanic abmhos", since the mechanic units are all referred, for convenience, to the C. G. S. system.

The size in Fig. 3 of the admittance Y or y, at any slope α deg., is $G \cos \alpha$ electrically, or $g \cos \alpha$ mechanically.

We may express the maximum cyclic angular displacement of the vibrator mirror by Θ radians $\angle$. The instantaneous angular velocity will then be $\dfrac{d\theta}{dt}$ or $\dot{\theta}$ radians per sec. and the maximum cyclic value of the mirror's angular velocity may then be denoted by $\dfrac{d\Theta}{dt}$, or $\dot{\Theta}$ radians per sec. Here $\dot{\Theta}$ corresponds to I.

Folliowng the same analogy with reference to (**6**),

$$\dot{\Theta} = F y \qquad \text{max. cyc. radians per sec.} \angle \quad (\mathbf{9})$$

where the v.m.t. F is taken as of standard phase, or zero slope. Consequently, Fig. 3 may be interpreted as the graph of angular velocity produced by an exciting current I max. cy. absamperes in the vibrator, at varied angular velocity, the scale of the diagram being suitably chosen. As ω varies from zero through ω_1, ω_0, and ω_2, to infinity, $\dot{\Theta}$ varies from $0 \angle 90$ deg., through $F y_1 \angle 45$ deg., $F g \angle 0$ deg., $F y_2$ ⦢ 45 deg. to 0 ⦢ 90 deg., radians per sec. $\angle$. This means that at the angular velocity of resonance, the angular velocity of the mirror is greatest, and is in phase with the exciting alternating current. At a very low frequency, $\dot{\theta}$ is insignificantly small, but in leading quadrature to the exciting current. At a very high frequency, $\dot{\theta}$ is again very small, but in lagging quadrature to the exciting current. At the quadrantal frequencies f_1 and f_2, the size of the angular velocity will be 0.707, or $2^{-1/2}$ of that at resonance, and will be in semiquadrature with the exciting current.

Thus in Fig. 3, interpreted electrically at 10^{-2} mho per unit of length, with impressed frequency $\omega = 10{,}000$ radians per second, or $u = 0.5$, $Y = 0.555 \times 10^{-2} \angle 56.3$ deg.; so that 1 volt max cy. e.m.f at this frequency and standard phase applied to the $C\,L\,R$ circuit, of the constants chosen above, would establish a current of $0.555 \times 10^{-2} \angle 56.3$ deg. max. cy. ampere, or 5.55 max. cy. milliamperes, leading the e.m.f by 56.3 deg. If, however, the mechanic application of Fig. 3 is considered, with 100 mechanic abmhos to the unit of length, one max. cy dyne-perp-cm. applied to the vibrator, at $\omega = 10{,}000$ or $u = 0.5$, would establish a max. cy. angular velocity of $55.5 \angle 56.3$ deg. radians per sec. As the mirror crossed the zero point on its scale, it would be moving with the angular velocity of 55.5 radians per second, or 3180 degrees per second, and this zero would be crossed 56.3 deg. ahead in phase of the impressed v.m.t.

In the electric system, resonance occurs at

$$\omega_o = \frac{1}{\sqrt{C\,L}} = \sqrt{S/L} \qquad \frac{\text{radians}}{\text{sec}} \qquad \textbf{(11)}$$

In the mechanic system, resonance correspondingly occurs at

$$\omega_o = \sqrt{s/m} \qquad \frac{\text{radians}}{\text{sec}} \qquad \textbf{(12)}$$

Damping Constant of Oscillating Systems, Electric and Mechanic. If we energize either the condenser or the reactor or both these elements in Fig. 1, and removing, from the mains, the branch circuit $C\,L\,R$, close it on itself, the damping factor of its current, assuming no loss of power in either S or L is known to be

$$\Delta = \frac{R}{2\,L} \qquad \text{hyps per sec.} \qquad \textbf{(13)}$$

This may be regarded as the hyperbolic angular velocity of decay in the circuit, expressible in hyperbolic radians per second. Similarly, if we energize either the wires, or the mass of the vibrator, or both these elements in Fig. 4, the damping factor of the angular velocity, assuming no loss of power in either s or m, is:

$$\Delta = \frac{r}{2\,m} \qquad \text{hyps per sec.} \qquad \textbf{(14)}$$

This is also the hyperbolic angular velocity of decay in vibration. This quantity Δ is very important in oscillographic theory. The reciprocal of Δ may be called the *oscillatory time constant* of either system, or

$$\tau_o = 1/\Delta = \frac{L}{(R/2)} = \frac{m}{(r/2)} \qquad \frac{\text{seconds}}{\text{hyp. radian}} \qquad \textbf{(15)}$$

This is equal to the time in which free oscillations fall to $1/\epsilon$ th in size. The damping factor may also be obtained from the quadrantal angular velocities, by the relation

$$\Delta = \frac{\omega_2 - \omega_1}{2} = \frac{f_2 - f_1}{\pi} \qquad \frac{\text{hyp}}{\text{sec.}} \qquad \textbf{(16)}$$

or from any angular velocity ω, at which the slope of the vector admittance is α deg., by the more general relation:

$$\Delta = \frac{\omega^2 - \omega_o^2}{2\,\omega \tan \alpha} \qquad \frac{\text{hyps.}}{\text{sec.}} \qquad \textbf{(17)}$$

In the case represented by Figs. 2 and 3, $\Delta = 10{,}000$ hyps. per sec.

Sharpness and Bluntness of Resonance. The sharpness of resonance of a circuit, or of a vibrator, may be defined as the ratio of the resonant angular velocity to the damping factor, or

$$\Lambda = \omega_o/\Delta = \frac{\omega_o}{\left(\frac{\omega_2 - \omega_1}{2}\right)} = \frac{2\,\omega_o\,\omega_1}{\omega_o^2 - \omega_1^2} = \frac{2\,\omega_o\,\omega_2}{\omega_2^2 - \omega_o^2}$$

$$= \frac{f_o}{\left(\frac{f_2 - f_1}{2}\right)} = \frac{2\,\pi\,f_o}{\Delta} = \frac{m\,\omega_o}{(r/2)} = \frac{s}{(r/2)\,\omega_o}$$

$$= \frac{\sqrt{m\,s}}{r/2} = \frac{\sqrt{L/C}}{R/2} \qquad \text{numeric} \qquad \textbf{(18)}$$

Very sharp resonance thus implies a large ratio of ω_0 to the semi-quadrantal range $\frac{(\omega_2 - \omega_1)}{2}$. Resonance sharpnesses in designedly resonant electric circuits are commonly of the first order, (near 10^1) and occasionally exceed the second order (10^2). In vibrators, the magnitudes encountered are of similar orders. In air-damped vibration galvanometers, a resonant sharpness as high as the third order, (1000) may be met.

The reciprocal of the sharpness of resonance may be called *B*, the bluntness of resonance

$$B = 1/\Lambda = \Delta/\omega_0 = \frac{\left(\frac{\omega_2 - \omega_1}{2}\right)}{\omega_0} = \frac{\omega_0^2 - \omega_1^2}{2\,\omega_0\,\omega_1}$$

$$= \frac{\omega_2^2 - \omega_0^2}{2\,\omega_0\,\omega_2} = \frac{\left(\frac{f_2 - f_1}{2}\right)}{f_0} = \frac{\Delta}{2\,\pi\,f_0} = \frac{(r/2)}{m\,\omega_0}$$

$$= \frac{(r/2)\,\omega_0}{s} = \frac{(r/2)}{\sqrt{m\,s}} = \frac{R/2}{\sqrt{L/C}} = \frac{u_1 - u_2}{2} \qquad \text{numeric} \qquad (19)$$

When the bluntness of a *C L R* circuit, or of a vibrator, attains the value unity, the system is critically damped, or is just aperiodic. If it exceeds unity, the system is overdamped, or ultraperiodic. If the bluntness is less than unity, or the sharpness greater than unity, free oscillations can occur. With high sharpness, numerous oscillations accompany oscillatory decay. Air-damped oscillographic vibrators commonly have a sharpness Λ of 100 or more, or a bluntness *B* of about 0.010 or less. Oil damped vibrators may easily have a bluntness *B* greater than unity. These numerical coefficients Λ and *B* are important in oscillographic theory.*

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*The resonance sharpness thus defined is double that used in some publications. See Bibliography 11, 13, 17. As defined in (**18**), it may be distinquished as Λ_0, and has certain algebraic advantages over the lesser value $\Lambda_s = \frac{\omega_0}{2\,\Delta}$. In this paper Λ stands for Λ_0, and its reciprocal *B* for B_0. Λ_0 may be regarded as an oscillatory sharpness, depending on the oscillatory time constant $\tau_0 = \frac{2\,L}{R}$ seconds, and as distinguished from the sharpness of sustained alternation Λ_s, depending on the time constant of continuous-current application $\tau_s = L/R$ seconds. Hence $B_s = 2\,B_0$, $\Lambda_s = \Lambda_0/2$; or $B_0 = B_s/2$ and $\Lambda_0 = 2\,\Lambda_s$.

In the case represented in Figs. 2 and 3, $\Lambda = 2$, and $B = ½$. This means, therefore, that in the case of those figures, the resistance to either electrical or mechanical motion is just half of that which would render the system aperiodic, or critically damped.

Rapidity of Change in Phase Angle α, of I or $\dot{\theta}$, with respect to ω. In sharply resonant systems, ($\Lambda > 1$), a small change in the impressed angular velocity will involve a rapid change in the phase angle α of current I, or angular velocity $\dot{\theta}$, Fig. 3, near resonance. In cases of very sharp resonance, or small damping, a change of a few radians per second in ω, near to ω_0, may carry the vector I, or $\dot{\theta}$, around the quadrantal range in slope (from $\alpha = +45$ deg. to $\alpha = -45$ deg.). On the other hand, in bluntly resonant systems ($B > 1$), it may require a relatively large change in ω to carry the vector I or $\dot{\theta}$ over the same range. In the neighborhood of the resonant angular velocity ω_0, we may notice that

$$\frac{d\,\alpha}{d\,\omega} = -\,1/\Delta = -\,\tau_0 \qquad \text{seconds} \qquad (\mathbf{20})$$

so that the phase α changes near resonance by $-\,\tau_0$ radians per single radian per second in ω. By (**14**), this change is rapid in a system of large L and small R, or of large m and small r. With a large frictional resistance r, this rate of change in α becomes small.

Again, the quadrantal range in α is from α_1 to α_2, or $-\,90$ deg. or $-\,\pi/2$ radians (see Fig. 3). The corresponding change in angular velocity is $\omega_2 - \omega_1 = 2\,\Delta = 2/\tau_0$. Consequently, the average rate of change in α with respect to ω, over the whole quandrantal range is

$$\frac{\alpha_2 - \alpha_1}{\omega_2 - \omega_1} = \frac{-\,\pi/2}{(2/\tau_0)} = -\,(\pi/4)\,.\,\tau_0 = -\,0.7854\,\tau_0 \qquad \text{seconds} \qquad (\mathbf{21})$$

or the average rate of change in α with respect to ω is 78.5 per cent, over the whole resonant range, of what it is at resonance, where the rate is a maximum. Above and below the quadrantal frequencies, at which the reactive factor

$$\frac{m\,\omega - s/\omega}{r} = \pm\,1,$$

the rate of change in α with change in ω diminishes. At large reactive factors, it diminishes approximately as the inverse square of the reactive factor.

Impedance to Displacement. The foregoing discussion relates to the evaluation of the maximum cyclic current I in a $C\ L\ R$ branch, or to the maximum cyclic angular velocity $\dot{\theta}$ of a vibrator mirror. This maximum cyclic angular velocity occurs at the moment when the mirror is passing through its zero or central position. The oscillographer is only indirectly interested, however, in the angular velocity of his mirror. He is directly interested in the angular displacements, or elongations, of his mirror. In the electric-circuit analogy, the mirror displacement corresponds to the alternating displacement of electricity, or electric quantity, Q coulombs $\angle$. The quantity Q is the time integral of the current I. In the assumed condition of a simple harmonic or purely sinusoidal current, the quantity Q is also sinusoidal. It is in direct proportion both as to magnitude and phase with the voltage across the condenser, or is in lagging quadrature with the current. We are therefore led to study the impedance to displacement or "displacement impedance" of a branch or a vibrator, in order to evaluate θ, the maximum cyclic angular displacement of a vibrator mirror.

It is easily shown that the displacement impedance Z', for the simple alternating electric circuit, is

$$Z' = j\,\omega\,\{ R + j\,(L\,\omega - S/\omega)\} = (S - L\,\omega^2) + j\,R\,\omega \qquad \frac{\text{volts}}{\text{coulombs}}\angle \qquad (22)$$

and similarly for the mechanical vibrator:

$$z' = j\,\omega\,\{r + j\,(m\,\omega - s/\omega)\} = (s - m\,\omega^2) + j\,r\,\omega \qquad \frac{\text{dyne perp. cm.}}{\text{radian}}\angle \qquad (23)$$

$$= s\,\{(1 - u^2) + j\,2\,B_o\,u\} = s\,\{(1 - u^2) + j\,B_s\,u\} \qquad \frac{\text{dyne perp. cm.}}{\text{radian}}\angle$$

As u varies from 0 to infinity, z' traces a parabola. When $B_0 = 0$, the parabola collapses into a straight line along the x axis. When $B = \infty$ it nearly coincides with a straight line parallel to the Y axis. Consequently, the max. cyclic displacement of electricity in the branch circuit, and accumulated in the condenser, from an impressed max. cyclic e.m.f. E volts $\angle$ on the branch, will be

$$Q = E/Z' \qquad \text{max. cy. coulombs } \angle \qquad (\mathbf{24})$$

Similarly, the max. cyclic displacement of the mirror of the vibrator, under a max, cyclic v.m.t. F dynes-perp.- cm., will be:

$$\theta = F/z' \qquad \text{max. cy. radians } \angle \qquad (\mathbf{25})$$

Fig. 5 is a graph of displacement impedance, for a case of $\Lambda = 2$, or $B = 0.5$. In such a case, the resistance r to motion is just equal to the surge resistance $\sqrt{m\ s}$, or

$$R = \sqrt{L/C} \qquad \text{ohms} \qquad (\mathbf{26})$$

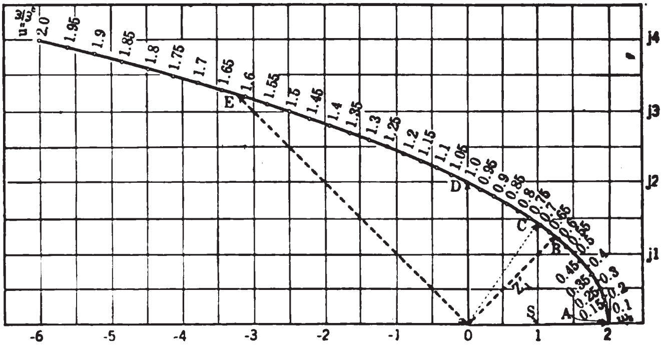

Fig. 5—Graph of Impedance to Electric or Mechanic Displacement for Cases of

$$\begin{Bmatrix} R = 10^2, & L = 5 \times 10^{-3}, & S = 2 \times 10^6 \\ r = 10^{-2}, & m = 5 \times 10^{-7}, & s = 2 \times 10^2 \end{Bmatrix}$$

$$w_0 = 20{,}000 \quad \Delta = 10{,}000 \quad B = 0.5$$

In the Electric Case, Unit Length Represents 10^6 Volts Per Coulomb.

In the Mechanic Case, Unit Length Represents 10^2 Dynes-Perp. Cm. Per Radian.

and the resistance is just 0.5 critical. In the current impedance graph of Fig. 2, similar relations between the ohmic and surge resistances were chosen; so that the graph of Fig. 5 may be derived from that of Fig. 2, according to (**21**), by applying to each vector, at angular velocity ω, the factor $j\ \omega$, which rotates the vector counter-clockwise through 1 quadrant and extends it in the numerical ratio ω, which ratio increases steadily as the frequency is increased. We thus rotate the diagram of Fig. 2 through + 90 deg., and apply the factor ω throughout.

It will be evident that at $\omega = 0$, the initial displacement impedance is S or s, as the case may be, taken as $O\,A$ Fig. 5, along the axis of real quantities. At the lower quadrantal angular velocity ω_1, this impedance is $O\,B$, at a slope of 45 deg. At resonance, or ω_o, this impedance has reached $O\,D$, along the j axis. At ω_2, it has reached $O\,E$, at the slope of 135 deg., and thence, as the angular velocity is increased indefinitely, the displacement impedance tends to an infinite size, at slope 180 deg. It will be observed that the curve of displacement impedance is a parabola $A\,B\,C\,D\,E$ Fig. 5, with its axis along the line of real quantities.

The displacement impedance is a minimum at that point C in the curve, where the tangent is perpendicular to the radius vector.

Displacement Admittance. The reciprocal of displacement impedance, or "displacement admittance," is of still greater interest and practical importance from the oscillographic viewpoint. Form (**23**), considering only the mechanical case, the displacement admittance is

$$y' = 1/z' = \frac{1}{(s - m\,\omega^2) + j\,r\,\omega} = \frac{1}{m\,(\omega_o^2 - \omega^2) + j\,r\,\omega}$$

$$= \frac{1}{m\,\{(\omega_o^2 - \omega^2) + j\,2\,\Delta\,\omega\}} \qquad \frac{\text{radians}}{\text{dyne perp. cm.}} \angle \qquad (\mathbf{27})$$

$$= \frac{1}{s\,\{(1 - u^2) + j\,2\,B_o\,u\}} = \frac{1}{m\omega_0^2\,\{(1 - u^2) + j\,2\,B_o u)\}}$$

$$= \frac{y}{j\,\omega} = -\,j\,y/\omega \qquad \frac{\text{radians}}{\text{dyne perp. cm.}} \angle \qquad (\mathbf{27a})$$

At $\omega = 0$, $y' = y_o' = 1/s \angle 0$ deg.

$$\frac{\text{radians}}{\text{dyne perp. cm.}} \angle \qquad (\mathbf{28})$$

At $\omega = \omega_1$, $y' = y_1' = \dfrac{1}{\sqrt{2}\,r\,\omega_1} \nwarrow 45$ deg.

$$\frac{\text{radians}}{\text{dyne perp. cm.}} \angle \qquad (\mathbf{29})$$

At $\omega = \omega_o, y' = y_o' = -j \dfrac{1}{r\ \omega_o} = \dfrac{1}{r\ \omega_o} \nwarrow 90$ deg.

$$\frac{\text{radians}}{\text{dyne perp. cm.}} \angle \qquad (30)$$

At $\omega = \omega_2, y' = y_2' = \dfrac{1}{\sqrt{2}\, r\ \omega_2} \nwarrow 135$ deg.

$$\frac{\text{radians}}{\text{dyne perp. cm.}} \angle \qquad (31)$$

At $\omega = \infty, y' = y'_\infty = 0 \nwarrow 180$ deg.

$$\frac{\text{radians}}{\text{dyne perp. cm.}} \angle \qquad (32)$$

Displacement Admittance Graph. Referring to (**27a**), and to Fig. 6, it will be seen that if we draw the circular graph of θ, $O\ B\ C\ D$, and at any particular value of ω, or $u\ \omega_o$, such as $\omega = 10{,}000$, draw the vector $O\ P$, of size 0.555×10^2 $\dfrac{\text{radians per sec.}}{\text{dyne perp. cm.}}$, at slope $\alpha = 56.3$ deg., we may then draw perpendicularly thereto the vector $O\ p$, and dividing 55.5 by ω obtain, to a correspondingly altered scale, the displacement admittance of 5.55×10^{-3} radian per dyne-perp.-cm., at a slope of β deg. $= -33.7$ deg. This means that if unit max. cy. v.m.t., or 1 dyne-perp.-cm., be applied to the vibrator of this case, at $\omega = 10{,}000$, the max. cy. displacement of the mirror will be 5.55×10^{-3} radians $= 0.32$ degree. This elongation, or max. cy. angular displacement, would occur 33.7 deg. in phase behind the corresponding elongation in torque or exciting current.

Proceeding in this manner, we might map out the entire displacement admittance* graph a, p, b, c, d, O. At or near zero frequency, denoted by ω_s, the displacement admittance would be the vector $O\ a$, in phase with the exciting current of the vibrator. At the lower quadrantal frequency ω_1, it would be the vector $O\ b$. At the upper quadrantal frequency, it would be $O\ d$. At resonance, it would be $O\ c$, in lagging quadrature with the excitation.

The maximum admittance would occur at ω_d, in this case at $u = 0.707$, or $\omega = 14{,}140$. This vector will vary in position with different values of B. *It is to be noted, however, that the maximum admittance never occurs at resonance, although in sharp vibrators, it will lie close to resonance.*

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*Bibliography 18, p. 505 and 511.

If we consider Fig. 6, in terms of its electric representation, the scale being 10^{-2} mho. per unit length, then at $\omega = 10{,}000$ and $u = 0.5$, the electric displacement admittance would be $0.555 \times 10^{-6} \measuredangle 33.7$ deg. coulomb per max. cy. volt on the $C\,L\,R$ branch terminals.

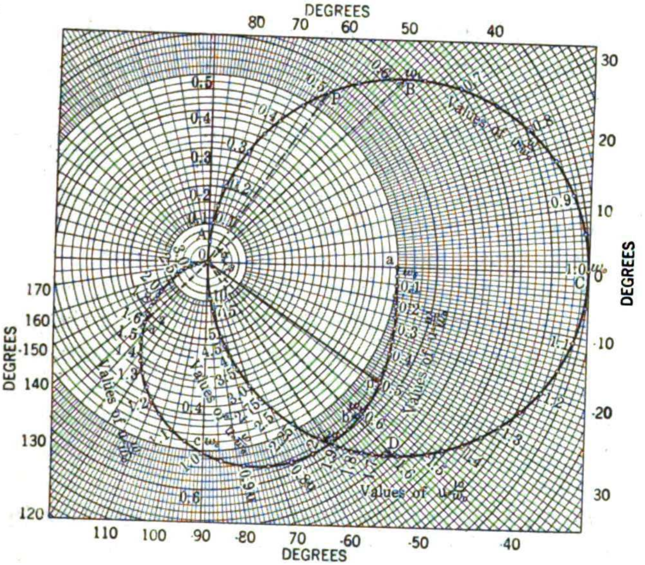

FIG. 6—GRAPH OF DISPLACEMENT ADMITTANCE FOR CASES OF

$$\left\{ \begin{array}{lll} R = 10^2, & L = 5 \times 10^{-3}, & S = 2 \times 10^6 \\ r = 10^{-2}, & m = 5 \times 10^{-7}, & S = 2 \times 10^2 \end{array} \right\}$$

$w_0 = 20{,}000 \quad \Delta = 10{,}000 \quad B = 0.5$

FOR CIRCLE $A\ B\ C\ D$ { IN THE ELECTRIC CASE, UNIT LENGTH $= 10^{-2}$ MHO.
IN THE MECHANIC CASE, UNIT LENGTH $= 10^{-2}$ MECHANIC ABMHOS, OR $10^2 \left(\frac{\text{RADIANS}}{\text{SEC.}}\right)$ PER UNIT V.M.T.

FOR GRAPH $a\ b\ c\ d$ { IN THE ELECTRIC CASE, UNIT LENGTH $= 10^6$ COULOMB PER VOLT.
IN THE MECHANIC CASE, UNIT LENGTH $= 10^{-2}$ RADIAN PER UNIT V.M.T.

It is also easy to show that the displacement admittance graph $a\ b\ c\ d$, Fig. 6, is obtainable, by direct vector inversion, from the displacement impedance graph *A B C D E*, of Fig. 5.

Displacement:

Since

$$\Theta = F\,y' \qquad \text{max. cy. radians } \angle \qquad (33)$$

and

$$Q = E\,Y' \qquad \text{max. cy. coulombs } \angle \qquad (34)$$

it is evident that the displacement admittance graph is also capable of being directly interpreted as a displacement graph, to constant impressed v.m.t. at standard phase, by adopting a suitable scale of linear dimensions. Thus, taking Fig. 6, at resonant frequency; *i. e.*, $u = 1.0$, and $\omega_o = 20{,}000$ radians per sec., or $f_o = 3183 \sim$, an impressed v.m.t. of say 10 max. cy. dynes-perp. cm. would produce a max. cy. mirror displacement of $\Theta = 0.05 \,\overline{\diagdown}\, 90$ deg. radian $= 2.87$ deg., in lagging quadrature with the excitation. A beam of light, reflected by the mirror, would be cyclically deflected to twice this angle or $\pm$ 5.74 deg., at elongation, on each side of the zero, owing to the doubling optical reflecting property of mirrors. It may be noted that, at resonance ($u = 1$), the displacement is just 90 deg. in phase behind the excitation and v.m.t., for any and all values of B.

Formulas for Displacement Admittance. Remarking that the slope β deg. of the displacement admittance Y' or y', is always negative, lying between the values $-\,0$ deg. and $-\,180$ deg., we have

$$\omega = \sqrt{\omega_o^2 + \Delta^2 \cot^2 \beta} + \Delta \cot \beta$$

$$= \omega_o \left\{ \sqrt{1 + B^2 \cot^2 \beta} + B \cot \beta \right\} \qquad \frac{\text{radians}}{\text{sec.}} \qquad (35)$$

$$\text{or} \quad u = \omega / \omega_o = \sqrt{1 + B^2 \cot^2 \beta} + B \cot \beta \qquad \text{numeric} \qquad (36)$$

The polar equation of displacement admittance is

$$| y' | = \frac{-\sin \beta}{r\,\omega} = \frac{-\sin \beta}{r\,\omega_o \left\{ \sqrt{1 + B^2 \cot^2 \beta} + B \cot \beta \right\}} \qquad \text{numeric} \qquad (37)$$

$$= \frac{-\sin \beta}{2\,B\,s \left\{ \sqrt{1 + B^2 \cot^2 \beta} + B \cot \beta \right\}} \qquad \text{numeric} \qquad (37a)$$

The corresponding vector equation is

$$y' = \frac{-\sin\beta}{r\,\omega} \angle -\ \beta \text{ deg.}$$

$$= \frac{-\sin\beta \ \bar{\diagdown}\ \beta \text{ deg.}}{2\,B\,s\ \{\sqrt{1+B^2\cot^2\beta}+B\cot\beta\}}$$

$$= \frac{-\sin\beta \ \bar{\diagdown}\ \beta \text{ deg.}}{2\,B\,s\,u} \qquad \frac{\text{mirror radians}}{\text{unit v.m.t.}} \angle \qquad (\mathbf{38})$$

The slope β, pertaining to any assigned value of ω, is

$$\beta = \tan^{-1}\left(\frac{2\,\Delta\,u}{\omega^2-\omega_o^2}\right) \qquad \text{degrees} \qquad (\mathbf{39})$$

$$\text{or}\quad \beta = \tan^{-1}\left(\frac{2\,B_o\,u}{u^2-1}\right) = \tan^{-1}\left(\frac{B_s\,u}{u^2-1}\right) \quad \text{degrees} \qquad (\mathbf{40})$$

In the testing of oscillographs, the calibration is ordinarily conducted at a relatively low frequency, say 60 ∼ , which, for practical purposes, may be regarded as of the same effect as zero frequency, or ω_s. By (**28**), the displacement admittance at ω_s is $1/s$, at zero slope. It is desired to find what ratio the admittance will bear at any impressed angular velocity ω, (or its ratio u), to that at the calibration frequency ω_s. From (**38**) and (**28**) we obtain

$$D = y'/y_s' = \frac{-\sin\beta \angle -\ \beta \text{ deg.}}{2\,B_o\,u} = \frac{-\sin\beta \ \bar{\diagdown}\ \beta \text{ deg.}}{B_s\,u}$$

$$= \frac{-\sin\beta \ \bar{\diagdown}\ \beta \text{ deg.}}{2\,B_o\{\sqrt{1+B_o^2\cot^2\beta}+B_o\cot\beta\}}$$

$$\text{numeric} \angle \qquad (\mathbf{41})$$

This is an important formula. It gives the vector *deviation factor* D, for any oscillograph at any impressed ω, when the values of ω_o and B are known for the instrument. Thus, in the case represented by Fig. 6, where ω_o is 20,000, and $B = 0.5$, if the instrument is calibrated at 60 ∼ $= \omega_s$, it may be required to know what will be the deviation factor at say $\omega = 5000$ or $u = 0.25$. Here, by (**40**),

$$\beta = \tan^{-1}\frac{0.25}{-0.9375} = \tan^{-1}(-0.2667) = -(14 \text{ deg. } 56 \text{ min.})$$

Then by (**41**)

$$D = y'/y_s' = \frac{-\sin(-14 \text{ deg. } 56 \text{ min.}) \;\overline{\diagdown}\; (14 \text{ deg. } 56 \text{ min.})}{2 \times 0.5 \times 0.25}$$

$$= \frac{0.2577}{0.25} \;\overline{\diagdown}\; 14 \text{ deg. } 56 \text{ min.} = 1.031 \;\overline{\diagdown}\; 14 \text{ deg. } 56 \text{ min.}$$

This means that the max. cy displacement produced at 796 ~, or 5000 radians/sec. will be 3.1 per cent greater than that produced by the same impressed torque at 60 ~, and will lag 14 deg. 56 min. in phase behind the impressed torque; whereas at 60 ~, the phase lag of displacement would be insignificant in such a vibrator. The *correction factor* will be the reciprocal of the deviation factor, or in this case 0.969 ∠ 14 deg. 56 min., at 796 ~. The frequency of 796 ~ would be 13 times 61.23 or would be the thirteenth harmonic of 61.23 ~. ~~The lag of mirror displacement to this harmonic would thus be nearly 15 deg. of fundamental frequency, or nearly 13 × 15 = 195 deg. of the harmonic frequency~~. The position of the recorded 13th harmonic would thus be retarded by nearly 195 deg. of its own cycle, and the harmonic, as recorded photographically should be advanced through more than ~~half~~ of its own cycle, in order to place it in proper relation to the fundamental. Its recorded or apparent magnitude should also be reduced 3 per cent, in order to have the same scale of amplitude as the fundamental and calibrated frequency.

The following Table of important displacement frequencies may be convenient for reference.

Of the above 8 important angular velocities, Nos. 1, 2, 5 and 7 are always presented, at the successive negative slopes of 0 deg., 45 deg., 90 deg., and 135 deg. No. 3, the maximum admittant value of ω, is always to be found when $B < 1$, or $\Lambda > 1$. For values of B equal to, or greater than, unity, which we may call blunt cases, ω_d has no independent existence, because the greatest admittance is then found at $\omega = \omega_s = 0$. Angular velocity No. 4, ω_f, is always presented when $B < 1$; but in blunt cases, the system is too heavily damped to permit of free vibration. No. 6 is always presented, but its phase position varies with B. It is midway in angular velocity between ω_1 and ω_2; but is not midway between them in phase. No. 8, the upper angular velocity at which the displacement has the same size as at ω_s, is not found when $B > 0.707$. In

TABLE I.
PRINCIPAL IMPRESSED ANGULAR VELOCITIES OF A VIBRATORY SYSTEM.

	Angular Velocity radians/sec.	Displacement Admittance Mirror radians/dyne p. cm.	
1. Initial, (near zero),	ω_s	$y_s' = 1/s \angle 0^\circ = c \angle 0^\circ$	(42)
2. Lower quadrantal,	$\omega_1 = \sqrt{\omega_0^2 + \Delta^2} - \Delta = \omega_0 \{ \sqrt{1 + B^2} - B \}$	$y_1' = \dfrac{1}{\omega_1 r \sqrt{2}} \angle 45^\circ = \dfrac{g}{\omega_1 \sqrt{2}} \angle 45^\circ$	(43)
3. Max. admittant,	$\omega_d = \sqrt{\omega_0^2 - 2\Delta^2} = \omega_0 \sqrt{1 - 2B^2}$	$y_d' = \dfrac{1}{r\,\omega_f} \angle \tan^{-1} (\omega_d/\Delta)$	(44)
4. Free vibrational,	$\omega_f = \sqrt{\omega_0^2 - \Delta^2} = \omega_0 \sqrt{1 - B^2}$	$y_f' = \dfrac{1}{r \sqrt{\omega_f^2 + (\Delta/2)^2}} \angle \tan^{-1} \dfrac{\omega_f}{(\Delta/2)}$	(45)
5. Resonant,	$\omega_0 = \omega_0^2 - 0 = \omega_0 \sqrt{1}$	$y_0' = \dfrac{1}{r\,\omega_0} \angle 90^\circ = \dfrac{-j\,g}{\omega_0}$	(46)
6. Midquadrantal,	$\omega_{12} = \sqrt{\omega_0^2 + \Delta^2} = \omega_0 \sqrt{1 + B^2}$	$y_{12}' = \dfrac{1}{r \sqrt{\omega_0^2 + 5\Delta^2/4}} \angle \tan^{-1} \left\{ \dfrac{\omega_{12}}{(\Delta/2)} \right\}$	(47)
7. Higher quadrantal,	$\omega_2 = \sqrt{\omega_0^2 + \Delta^2} + \Delta = \omega_0 \{ \sqrt{1 + B^2} + B \}$	$y_2' = \dfrac{1}{r\,\omega_2 \sqrt{2}} \angle 135^\circ = \dfrac{g}{\omega_2 \sqrt{2}} \angle 135^\circ$	(48)
8. Duplicate initial,	$\omega_{s2} = \omega_d \sqrt{2} = \omega_0 \sqrt{2(1 - 2B^2)}$	$y_{s2}' = c \angle \tan^{-1} \left(\dfrac{2\Delta\,\omega_{s2}}{\omega_d^2 - 2\Delta^2} \right)$	(49)

sharp cases, it is always present. Its phase position, or slope, varies with B.

A displacement admittance graph presenting all of these 8 angular velocities is seen in Fig. 7, for the particular case of $\omega_0 = 25{,}000$, or $f_0 = 3979 \sim$, and $B = 0.225$. Here y_s' is taken as 1, corresponding to an oscillographic standard calibration at or near zero frequency. We find y_1, y_0 and y_2, all at their corresponding slopes of -45 deg., -90 deg. and -135 deg., with the respective sizes of 1.9642, 2.22 and 1.2571, at the frequencies 3182, 3979 and 4973 $\sim$. The maximum admittance value y', at $\omega_d = 23{,}701$, or $f_d = 3{,}772 \sim$, is 2.2807, occurring at $\beta = -76$ deg. 38 min. 55 sec. The free vibrational value y_f', at $\omega_f = 24{,}359$, or $f_f = 3{,}877 \sim$, is 2.2657 $\angle$ 83 deg. 24 min. 50 sec. The midquadrantal value y_{12}, at $\omega_{12} = 25{,}625$, or $f_{12} = 4078 \sim$, is 2.1551 $\angle$ 96 deg. 15 min. 49 sec. Finally, the duplicate initial value of y', or y_{s2}', at $\omega = 33518$, or $f_{s2} = 5334 \sim$, is 1.0 $\angle$ 142 deg. 53 min. 30 sec. In the oscillograph corresponding to this case, the deviation factor at $\omega = 4000$, or $f = 636 \sim$, would be 1.023 $\angle$ 4 deg. 13 min. and at $\omega = 8000$, or $f = 1272 \sim$, it would be 1.101 $\angle$ 9 deg. 7 min.

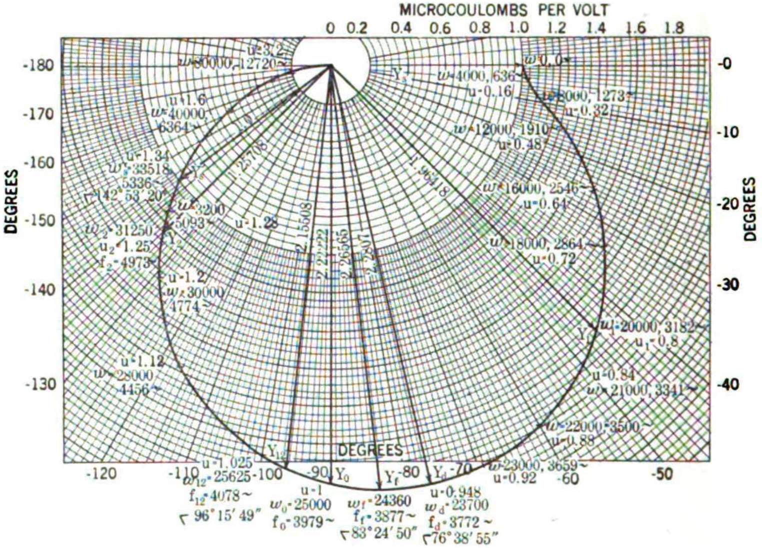

FIG. 7—VECTOR DISPLACEMENT ADMITTANCES

$w_0 = 25{,}000 \quad \Delta = 5{,}625 \quad B = 0.225 \quad s = 10^6 \quad m = 0.0016 \quad r = 18$

Fig. 8 is a vector chart of the deviation factor D for any value of bluntness B between $B = 0.25$ and $B = 1.0$. Enter the chart with u, the ratio of the impressed angular velocity; to the known resonant angular velocity ω_o. The intersection of the u line and the B line will lie on a radius vector whose size and slope make the factor D.

Fig. 9 is a similar vector chart for finding D, when B lies between 1.0 and ∞. There is actually but little change in the form of the curve for values of B exceeding 4. The limiting curve is a semi-circle of unit diameter on the real axis.

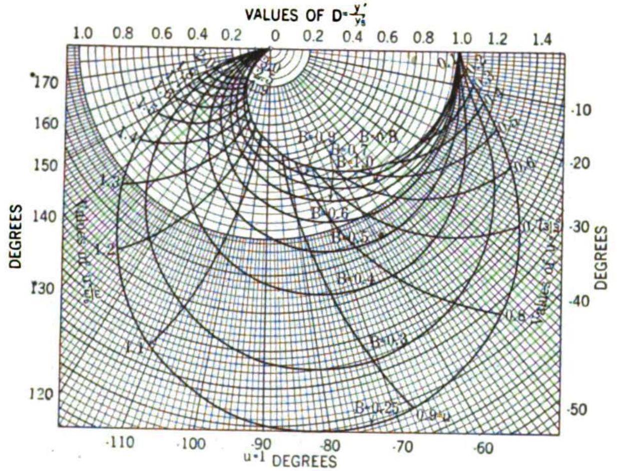

Fig. 8—Planevector Chart of the Deviation Factor D for any Oscillograph whose Bluntness Lies between $B = 0.25$ and $B = 1.0$, and whose Resonant Frequency is Given.

Example: If $B = 0.6$, and $u = 0.3$, then $D = 1.022 \,/\overline{21.6 \text{ Deg.}}$

Fig. 10 is a similar vector chart for D, when B lies between 0.05 and 0.25 inclusive. As B diminishes, the curve of D rapidly approximates to a circle of diameter $\frac{1}{2B} = \Lambda/2$, or radius $\frac{1}{4B}$, and having its center displaced approximately 0.25 unit to the right of the axis $-O\gamma$. At $B = 0$, this circle has an infinitely great diameter.

Constants of an Oscillograph Vibrator. It may be seen from an inspection of (**39**), (**40**) and (**41**), that the deviation factor of any oscillograph vibrator can be either computed or identified from the foregoing curve sheets, if we can ascertain its two essential working constants ω_o and Δ; or what is equivalent thereto, ω_o and B, or f_o and B. The tests of oscillographs thus direct themselves towards the determination of these two constants. If, however, not merely the deviation factor of a vibrator; but also its essential elements are required; then we must seek for the four constants A, m, r and s. This calls, as we shall see, for an additional test.

If we divide (**46**) by (**42**), we find:

$$y_s'/y_o' = j\,2\,B_o = 2\,B_o \angle\, 90 \text{ deg.} = B_s \angle\, 90 \text{ deg.} \qquad \text{numeric} \angle \qquad (\mathbf{50})$$

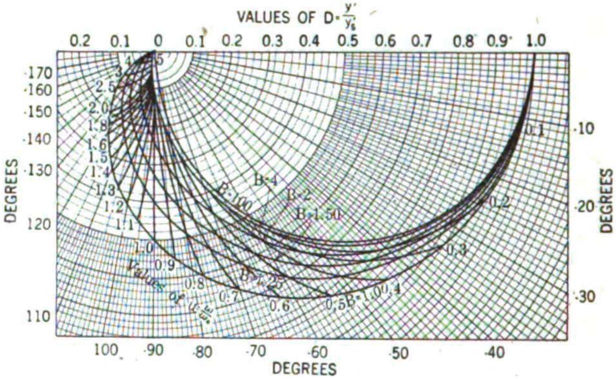

Fig. 9—Planevector Chart of Deviation Factor $D = y'/y'_s$ for any Oscillograph whose Bluntness Lies between $B = 1$ and $B = \infty$, and whose Resonant Frequency is Given.

Example: If $u = 0.5$ and $B = 2.0$, then $D = 0.468 \,/\, \overline{69.4 \text{ Deg.}}$

If, therefore, we can (1) identify the resonant frequency f_o, of a vibrator, and (2) compare the calibration of the vibrator at this frequency with that at zero frequency, we shall have; see Figs. 6 to 10:

$$\left|\frac{y_s'}{y'_o}\right| = 2\,B_o = B_s \qquad \text{numeric} \qquad (\mathbf{51})$$

The bluntness B_o of the vibrator will then be half the ratio of the max. cy. displacement at zero frequency to the max. cy. displacement at resonant frequency, equal current excitations and impressed torques being used in each test. If the vibrator is sharp, the same fact may be expressed by saying that *the sharpness Λ_o will be twice the ratio of the electric exciting* *current needed to produce the same max. cy. displacement, or deflection, at zero frequency, as will produce it at resonant frequency.*

All that is necessary, therefore, for finding the deviation factor of a vibrator at any impressed frequency, is to identify its resonant frequency f_o, and then compare its calibration at that frequency with that at f_s, a low frequency, practically equivalent to zero.

An equivalent alternative formula is:

$$\Delta = \frac{\omega_o}{j^2} \cdot \frac{y_s'}{y_o'} = \pi f_o \frac{y_s'}{y_o'} \nwarrow 90 \text{ deg. } \frac{\text{hyps}}{\text{sec.}} \angle \qquad \textbf{(52)}$$

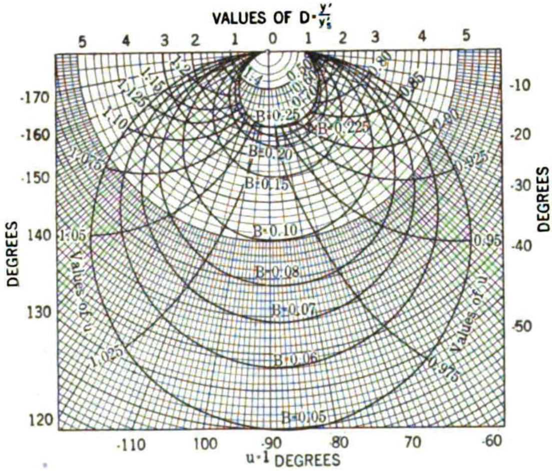

Fig. 10—Planevector Chart of the Deviation Factor $D = y'/y'_s$ for any Oscillograph whose Bluntness Lies between 0.05 and 0.25, and whose Resonant Frequency Is Given.

Example: If $u = 0.9$, and $B = 0.08$, $D = 4.17 \,/\overline{37.1 \text{ Deg.}}$

Oscillographmeter. In order to detect and identify the resonant frequency f_o of a vibrator, which, as above pointed out, is not the same as, and may even be remote from, the frequency f_d of maximum displacement, an oscillographmeter has been devised and constructed. It is shown separately in Fig. 11, and mounted in front of an oscillograph under test in Fig. 11-a. It consists of a simple permanent-magnet auxiliary vibrator, air-damped, and therefore of very sharp resonance. This vibrator is firmly set with its vibration axis horizontal, and with its mirror close to that of the tested oscillograph mirror, whose vibration axis is vertical. A strong beam of light is then directed first on the oscillograph mirror, thence on to the auxiliary mirror, and thence vertically upwards to a horizontal sheet of thin graduated paper; so that the path of the vibrating beam on this paper can be examined. The optical arrangement is indicated in Fig. 12.

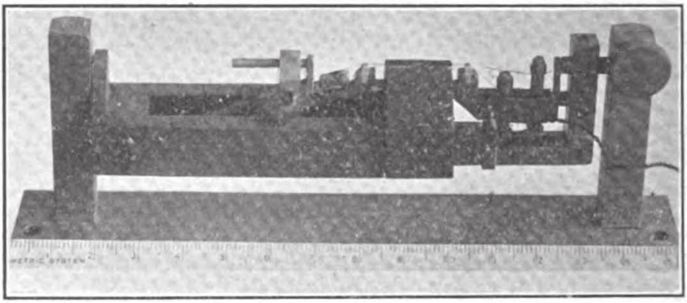

FIG. 11—OSCILLOGRAPHMETER OR AUXILIARY VIBRATOR. LENGTH OF MIRROR 2.3 MM. BREADTH 0.9 MM.

If the oscillograph and the auxiliary vibrator are operated from the same audio-frequency source of adjustable frequency; then they will both vibrate with that frequency. Because their vibration axes are set mutually perpendicular, the

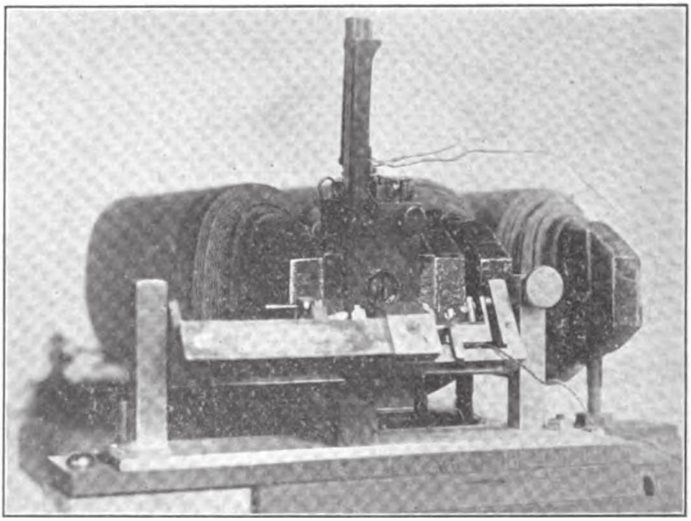

FIG. 11-A—OSCILLOGRAPHMETER APPLIED TO OSCILLOGRAPH FOR TEST OF LATTER

vibratory path of the beam of light on the paper plane *F F*, Fig. 12, will be a Lissajous figure*. If the max· cy. displacements of the two mirrors are cophasal, the figure formed by the moving beam will be a straight line. If they are not cophasal, the figure will be an ellipse. If they are in quadrature, this ellipse will include its maximum area and its two perpendicular axes will be both equal and parallel to the two component displacements. If these two components are equal, the ellipse will, at this quadrature phase relation, become a circle.

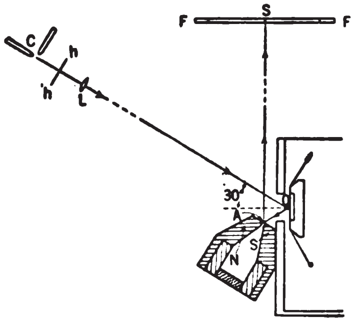

Fig. 12—Optical System of Oscillographmeter *A* and the Tested Oscillograph *O*. Length of *A* Mirror 2.3 mm. Width 0.9 mm. Distance *O–A* = 1.1 cm. Distance *A–S* = 50 cm.

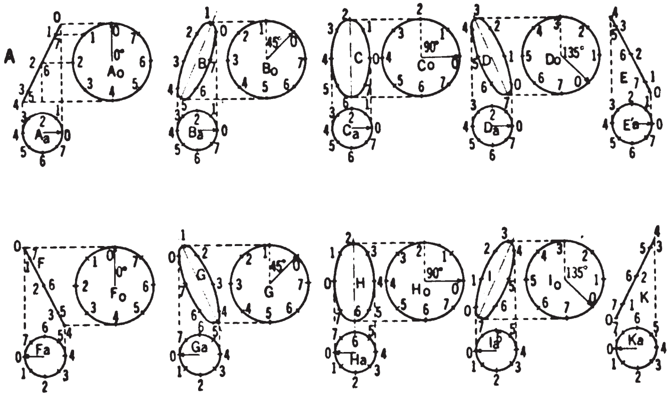

Fig. 13—Lissajous Figures with Components in the Ratio of 2 to 1 and Various Phase Differences

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*Bibliography 1.

The development of Lissajous ellipses is outlined in Fig. 13, for the particular case in which the auxiliary vibrator amplitude is half of the oscillograph vibrator amplitude. Commencing at A, the auxiliary vibrator A_a, starts at standard phase, and the oscillograph A_o, set in mechanical quadrature therewith, also at standard phase. The two vectors then rotate with equal angular velocities. The path of the reflected beam will be straight line A; *i. e.*, an ellipse of negligibly small minor axis.

When the oscillograph vibrator is lagging 45 deg. in phase with respect to the auxiliary vibrator, the path of the beam of light on the plane of the receiving sheet, will be the ellipse B. Again, with the oscillograph vibrator 90 deg. behind C_a in phase, the path of the beam will be the upright ellipse C. With 135 deg. of lag, the ellipse will incline backwards as at D. With 180 deg. of lag, the ellipse will collapse into the straight line E.

The effect of a reversal of phase in either the oscillograph or the auxiliary vibrator, is shown at F, G, H and K. In each case it will be seen that, at quadrature, the ellipse stands upright. The direction of change in inclination of the ellipse is, however, relatively reversed in the two series.

In making the test of the oscillograph, we must first make sure that the resonant frequency of the auxiliary vibrator is not near to the resonant frequency of the oscillograph vibrator. If the two resonant frequencies should happen to approach too closely, the tension of the auxiliary vibrator must be mechanically adjusted, in either direction, so as to separate them. All that is required of the auxiliary vibrator is that it shall be sharp, and also out of tune with the tested oscillograph. The various constants ω_o and B, or ω_o and Δ, or A, m, r and s, of the auxiliary vibrator are of no concern, provided that its B is very small, and that its ω_o is not close to the ω_o of the oscillograph vibrator.

Connections for Test. The electrical connections for the test of an oscillograph by means of the oscillographmeter, or auxiliary vibrator, are indicated at A, Fig. 14. S is the source, or secondary coil of an oscillator, supplying, in parallel, both the oscillograph O, and the auxiliary vibrator A, each through an adjustable resistance. The two supply currents will thus have the same phase, and the two impressed v.m.ts. will be cophasal. The Lissajous figure, obtained from the optical combination of the two vibrators, will then be a straight line, as at A, Fig. 13, if the impressed frequency is low, and remote from resonance in either vibrator. The impressed frequency is now raised, until the resonant angular velocity ω_0, or resonant frequency f_0, for O, is approached. The Lissajous ellipse will now begin to standup. The impressed frequency is now carefully adjusted, until the Lissajous ellipse is upright, and the frequency then noted. If the two vibratory components are equal, the Lissajous ellipse, at resonance in O, will be a circle, as shown at *A*, Fig. 14. This, however, is an unsuitable adjustment, since it does not lend itself to precise observation. An ellipse of say 2 : 1, or 3 : 1, in major and minor diameters is preferable, as at *C*, Fig. 13.

If the optical adjustment is fine, so that the beam of light is sharp and thin, it may be more easy to observe the existence of the thin cophase straight line of *A*, *E*, *F* and *K*, in Fig. 13, than

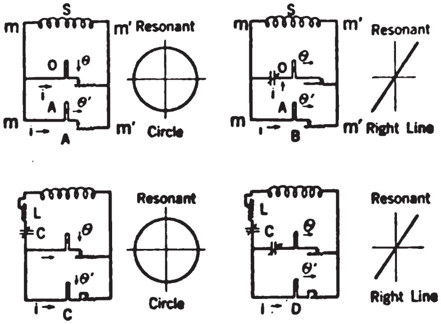

Fig. 14—Method of Testing Oscillograph Vibrators by Means of Oscillographmeter

the verticality of an ellipse. This is a matter of personal choice with the observer. If he prefers the straight line, or collapsed ellipse, then the connections of Fig. 14 B, may be used. Here, the resistance in one of the two vibrator circuits, O as shown, is replaced by an adjustable condenser. This has the effect of advancing the phase of the exciting current in that branch by 90 deg., so that the two impressed v.m.ts. will be in quadrature. If the two vibrator displacement slopes are cophasal, the Lissajous figure will then be the circle, or upright ellipse. At displacement slope quadrature, on the contrary, the two actual displacements will be cophasal, and resonance in O will show itself by the Lissajous straight line of Fig. 14B .

With the auxiliary vibrator air damped, and reasonably well constructed, its bluntness *B* will be very small, and its displacement slope will then be negligible, until within a few cycles per second of its resonant frequency. This means that the displacement slope of the auxiliary vibrator will either be substantially 0 deg., or substantially 180 deg., provided that the two vibrators are not closely coresonant.

It has been found that when a Vreeland oscillator is used as the source of impressed frequency, difficulties may arise owing to small parasitic harmonics or impurities in the derived current wave, which may give rise to false resonances, and erroneous results, in the behavior of the sharply resonant auxiliary vibrator. Much trouble was experienced in the first few months of this research, by such parasitic resonances. They can be eliminated, however, by electrically tuning the main testing circuit to the impressed frequency, as indicated in Fig. 14 at *C* and *D*. An adjustable inductor and condenser are so varied, that their combined reactance is negligibly small to the main impressed frequency from the source; while opposing a very considerable reactance either to parasitic frequencies, or harmonics. If this plan is used, care must be taken to make the proper adjustment of reactance at each change of impressed frequency. With the ordinary connections of a pliotron, or triode vacuum-tube oscillator, this difficulty has not been encountered.

Technique of Test. Having identified the resonant frequency f_o, or resonant angular velocity ω_o, of the oscillograph vibrator, with the aid of the oscillographmeter, and suitable Lissajous optical figure, the oscillograph is calibrated for its specific deflection, or maximum cyclic deflection per unit of r.m.s. exciting current, both at a low impressed frequency, f_s, say 60 ∼, and at f_o. In this part of the test, the oscillographmeter is either removed, or left out of circuit. As we have already seen in connection with (**51**), the ratio of these two calibrations, or specific deflections, is equal to the value of 2 B for the instrument. Having found ω_o and B, we are able to find the vector deviating factor, and its reciprocal, the correcting factor, for the instrument, at any or all impressed frequencies that may be used. Defining the specific deflection θ as the vector scale deflection of the oscillograph divided by the r.m.s. value of the exciting current in the oscillograph vibrator, we have, if θ_s is the specific deflection at low frequency, and to standard phase

$$D = \theta/\theta_s = \frac{1 - u_s^2 + j\,2\,B_o\,u_s}{1 - u^2 + j\,2\,B_o\,u} = \frac{1 - u_s^2 + j\,B_s\,u_s}{1 - u^2 + j\,B_s\,u} \quad \text{numeric } \angle \qquad (\mathbf{53})$$

Since the ratio ω_s/ω_o is usually a negligibly small real number, this may be written without serious error

$$D = \theta/\theta_s = \frac{1}{1 - u^2 + j\,2\,B_o\,u} = \frac{1}{1 - u^2 + j\,B_s\,u}$$

$$= \frac{-\sin\beta \angle -\beta}{2\,B_o\,u} = \frac{-\sin\beta \nwarrow \beta}{B_s\,u} \qquad \text{numeric } \angle \quad (\mathbf{54})$$

which agrees with (**50**) and (**51**), at resonance, when $u = 1$.

If we seek only the size of this ratio, and ignore the slope or phase deviation, we may write this in the form

$$|D| = \left|\frac{\theta}{\theta_s}\right| = \frac{1}{\sqrt{1 + 2\,u^2\,(2\,B_o^2 - 1) + u^4}} \qquad \text{numeric} \quad (\mathbf{55})$$

which is devoid of imaginary quantities. The correcting factor is then

$$K = 1/D \qquad \text{numeric} \quad (\mathbf{56})$$

When making specific deflection tests with the optical arrangement of Fig. 12, it has been found advantageous to insert a suitable lens L, with a fine cross hair on it, into the path of the beam. The positions of the hair can then be detected at each end of the vibrating beam of light on the graduated paper scale $F\,F$. When, however, Lissajous figures are being examined, the lens and crosshair are withdrawn.

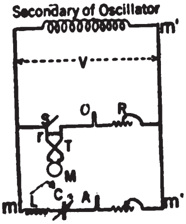

FIG. 15—METHOD OF DETERMINING QUADRANTAL POINTS

It should be remembered that the angular deflection of the beam of light reflected from the oscillograph's mirror is double the angular deflection of the mirror. Since the equations refer to the angular displacements of the mirror, this factor of 2 should be taken into account in interpreting the deflections.

Alternative Methods. Instead of determining B by the ratio of specific deflections θ_s and θ_o, it is possible, after finding ω_o, to find Δ, and hence B, by identifying optically one, or preferably both of the quadrantal angular velocities ω_1 and ω_2. The electrical connections for this are shown in Fig. 15. The plan is to introduce such equal resistance and reactance into the auxiliary branch circuit, as shall cause the exciting current and v.m.t., in one circuit, to differ in phase from that in the other by

45 deg., and so to enable quadrantal frequency to be identified optically from a Lissajous diagram. After observing ω_o, and then say ω_1, we can compute ω_2 by (**6**). Again, observing all three values ω_1, ω_o and ω_2, they should check by (**6**). We then find Δ by (**16**), and *B* by (**19**).

This alternative method enables a calibration of the oscillograph at resonant frequency to be dispensed with. On the other hand, it is apt to be less precise than the preferred method; because Δ is found as the difference of two identified frequencies. With sharp vibrators, these two quadrantal frequencies lie close together, and their difference is thus subject to relatively considerable error. In complete and thorough tests of an oscillograph, however, it may be worth while identifying all three values ω_o, ω_1, and ω_2, besides finding θ_s and θ; so as to secure mutual checks.

Another method for finding Δ and ω_o, without the use of the oscillographmeter, when the vibrator is sharp, is to make an oscillogram of a series of naturally decaying oscillations of the mirror, impressing on the same record a known fairly high frequency, from the movement of a second vibrator separately excited. The rate of decay in amplitude is given by the relation

$$\Theta = \Theta_i \epsilon^{-\Delta t} \qquad \text{max. cy. radians} \qquad (\mathbf{57})$$

where Θ_i is the initial displacement, at $t = 0$, and Θ the displacement at time t seconds

whence $$\Delta = (1/t) \operatorname{logh} \Theta_i/\Theta \qquad \text{hyps. per sec.} \qquad (\mathbf{58})$$

The frequency of vibration, as determined from the record, is

$\dfrac{\omega_f}{2\pi}$ cycles per second.

This method can be applied with fair success, to sharp vibrators. It cannot ordinarily be applied when *B* is greater than 0.2. This means that in ordinary liquid-damped oscillographs, this alternative is unsatisfactory.

It might be supposed that since a calibration is desired near to zero frequency, a continuous-current calibration would suffice. It has been found, however, that specific deflections with continuous currents are considerably in excess of those measured with low-frequency alternating currents, apparently because the elastic system has a greater opportunity to stretch under a steady impressed torque than under alternating torques.

A fairly typical and commercial form of oscillograph has its behavior represented in Fig. 16. This oscillograph is damped by a specially prepared liquid, supplied with the instrument. By means of the oscillographmeter, and Lissajous figures, the following frequencies are identified, $f_1 = 2663 \sim$, $f_o = 3300 \sim$, $f_2 = 4077 \sim$. The specific deflection θ_o, at resonance, was observed to be 5.35 mirror radians per r.m.s. absampere. The specific deflection θ_s at 60 ~, was also observed to be 2.25 mirror radians per r.m.s. absampere. The ratio θ_o/θ_s is 2.38; so that $\Lambda = 4.76$ or $B = 0.21$. From ω_o and B, the curve of Fig. 16 has been drawn, by the use of (**40**) and (**41**) or of (**54**). In addition, a number of values of the specific deflection were observed experimentally at different frequencies. These are marked as crosses on the diagram. If these observations were in complete accordance with the theory, the crosses should coincide with the curve. It will be noted that θ_d has the value 5.4 mirror radians per absampere at 3140 ~, which is 160 ~ below f_o. In this instrument, at 1000 ~, the size of the deviation factor is 1.09, or the instrument overindicates 9 per cent at 1000 ~, with respect to the low-frequency calibration. The slope of the deviating factor is also $\beta = -8$ deg. by (**40**).

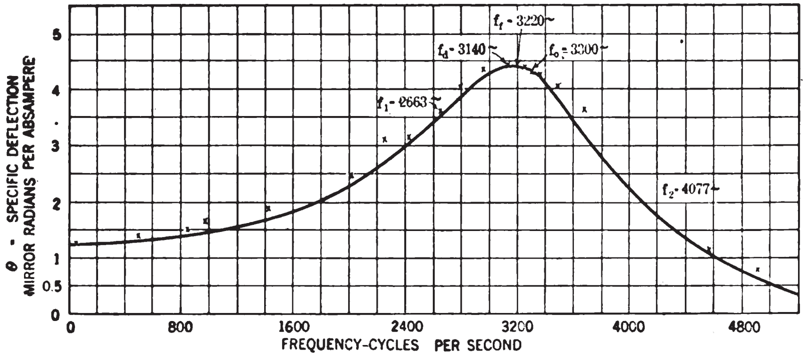

Fig. 16—Specific Deflection vs. Frequency

Oscillograph Vibrator in Light Damping Liquid. Δ = 4450 hyps. per second.

It may be observed that a curve of specific deflection θ as ordinates, against frequency f as abscissas, always rises to a maximum at f_d, unless B exceeds unity. The curve is always dissymmetrical about the maximum ordinate, being steeper on the high-frequency side. But if the curve of θ as ordinates be drawn against frequency squared, or f^2, as abscissas, the curve will be symmetrical about the maximum ordinate, as far as f_2, *i. e.*, as far as symmetry is possible.

Fig. 17 gives the curves of specific deflection θ, against impressed frequency f, for a vibrator in four successive damping fluids; *viz.*, air, glycoline, mineral oil and castor oil. The results are as follow:

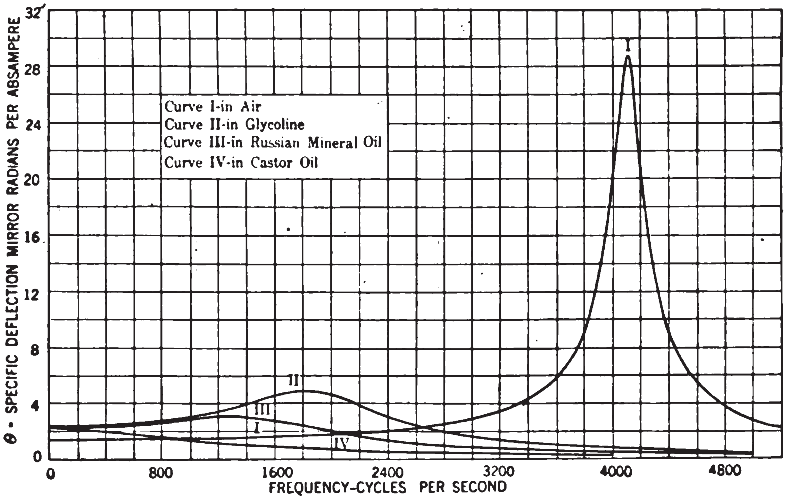

FIG. 17—SPECIFIC DEFLECTION VS. FREQUENCY
Oscillograph Vibrator 0-6. Temp. 26.5 deg. cent.

TABLE II

CONSTANTS OF AN OSCILLOGRAPHIC VIBRATOR IN DIFFERENT DAMPING FLUIDS

Curve	Damping fluid	f_0 cycles/sec	ω_0 rad./sec.	B_0 numeric	f_1 cycles/sec.	f_2 cycles/sec.	Δ hyps/sec.	f_d cycles/sec.	Resonant range cycles/sec.
I	Air	4,110	25,820	0.024	4,013	4,207	609	4,110	194
II	Glycoline	1,891	11,880	0.26	1,523	2,513	3,100	1,827	490
III	Mineral Oil	1,614	10,140	0.43	1,060	2,460	4,400	1,272	1,400
IV	Castor Oil	1,302	8,180	1.1	505	3,365	8,980	0	2,860

Whereas in air, the vibrator had a bluntness of 0.024, or a sharpness of 41.7, in castor oil, the bluntness was increased to 1.1 or a sharpness of 0.9. The effect was also to change the resonant frequency f_0 from 4110 ~ to 1302 ~.

In case III, the resistance r of the vibrator to motion is 0.43 of that which would render the motion aperiodic. In case IV, however, the resistance is ten per cent greater than that necessary for strict aperiodicity. The case is therefore one of over-damping, or ultraperiodicity. It is developed in greater detail in Fig. 17. The curve is drawn from the formulas already given, and the crosses indicate the observations. It will be seen that $f_d = 0$. At resonance, the specific deflection is only about half that at zero frequency.

It may be noted that except in the air-damped case, the value of θ_s is substantially the same for all the damping fluids. This condition seems to be typical.

A considerable number of oscillographs have been investigated by means of the oscillographmeter, and the general results are typically represented by Figs. 15 to 17. It appears that a bluntness of about 0.6 is perhaps the most desirable, in a long-range oscillograph, in the sense that the deviation factor is then moderate, over a considerable range of impressed frequency. Even then, however, the constants ω_o and B should be measured, and the correcting factor worked out, whenever accurate oscillographic measurements are attempted. In general, an oscillograph should have a bluntness between $B = 0.5$ and $B = 0.8$, in order to avoid large correcting factors.

Tests of the Auxiliary Vibrator. An exploration test of θ versus f for one of the oscillographmeters used in this research, is graphed in Fig. 18. It was also tested by the oscillographmeter method. The vibrator is seen to be very sharp. The three principal frequencies are $f_1 = 2511 \sim$, f_o 2514.5 $\sim$, f_2 2518 $\sim$. The resonant range is 7 cycles per second, and the bluntness $B_o = 0.0014$, $\Lambda_o = 718$. In order to present the peak value of θ on the same sheet as the earlier values, the observations are plotted to two different scales. It is evident that provided the resonant frequency is slightly avoided, the phase departure of the auxiliary mirror from zero or 180 deg. in the v.m.t. will be small.

Precautions to be taken. In supporting the auxiliary vibrator in front of the oscillograph, it is important that the auxiliary steel frame does not come so close to the poles of the oscillograph magnet, as materially to disturb the magnetic field in which the oscillograph vibrator swings. At the same time, the auxiliary and oscillograph mirrors have to be brought within a distance of about 1.3 cm., in order to secure good Lissajous figures, without making the auxiliary mirror unfavorably long. If the magnetic field of the oscillograph is somewhat weakened, this will not appreciably affect the identifiable frequencies f_1, f_o, and f_2. It may affect θ_s and θ_o, however; so that where there is reason to believe that the oscillograph's magnetic field has been affected by the proximity of the oscillographmeter, the specific deflections θ_s and θ_o should be measured with the oscillographmeter removed.

It should be noted that any dissymmetry in the construction of a vibrator, such as unequal dimensions, or unequal tensions in the two strips composing it, or a dissymmetrical mounting of the mirror, may give rise to plural resonances in the action of the instrument. A number of such cases have been observed. A good airdamped oscillographic vibrator requires good mechanical workmanship. When the vibrator is liquid damped these plural resonances often disappear.

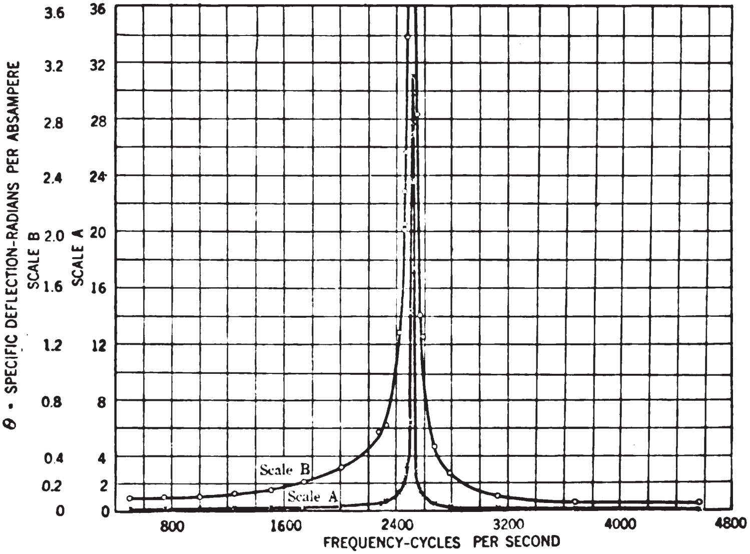

FIG. 18—SPECIFIC DEFLECTION VS. FREQUENCY
Auxiliary Vibrator A-1. Temp. 22.5 deg. cent.

Measurements Necessary for Evaluating A, m, r and s. In order to evaluate the fundamental constants of an oscillograph vibrator, it seems necessary to obtain an additional independent datum, such as is supplied by measuring the motional impedance of the vibrator at resonant frequency. This motional impedance is a pure resistance, and, unlike that of the telephone receiver, has no reactive component at resonant frequency. It is, however, relatively very small. Whereas the resonant motional impedance of a telephone receiver is sometimes over 1000 ohms, and is often over 100 ohms, that of a vibration galvanometer near $f_o = 1000 \sim$ may be near 10 ohms*; that of an ordinary liquid-damped oscillograph is in the neighborhood of 0.025 ohm. Airdamped oscillographs have larger motional impedances.

If we call Z the measured motional impedance of the oscillograph vibrator, at resonant frequency, expressed in absohms, and θ_m the observed max. cy. mirror displacement at resonant frequency per max. cy. absampere; then*

$$A = \frac{Z}{\theta_m \omega_o} \qquad \frac{\text{dynes perp. cm.}}{\text{absampere}} \angle \qquad (59)$$

$$r = \frac{A}{\theta_m \omega_o} \qquad \frac{\text{dyne perp. cm.}}{\text{mirror radian per sec.}} \qquad (60)$$

$$m = \frac{r}{2\,\Delta} \qquad \text{gm.} - \text{cm.}^2 \qquad (61)$$

$$s = m\,\omega_o^2 \qquad \frac{\text{dyne perp. cm.}}{\text{mirror radian}} \qquad (62)$$

An analysis of several oscillographic vibrators is given in Table III.

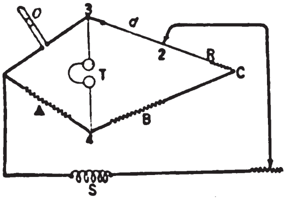

FIG. 19—SLIDE-WIRE BRIDGE
For measuring the motional impedance of an oscillograph vibrator O

Technique for Measuring Motional Impedance. A convenient form of slidewire Wheatstone bridge is indicated in Fig. 19, for the purpose of measuring the small motional impedance of a liquid-damped oscillograph. Two equal anti-inductive resistances A and B have about one ohm each. The oscillator, supplying the adjustable impressed frequency, is brought to slider 2 and junction 1. The slider 2 can be adjusted to a fraction of a millimeter, over a manganin slide wire, having a linear resistance not exceeding 0.01 ohm per cm. A pair of low-impedance head telephones T, enables a re-

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*Bibliography 13.
*Bibliography 13.

TABLE III. DATA FOR SOME OSCILLOGRAPH VIBRATORS TESTED

Vibrator Damping fluid	V_1 Glycoline	V2 Glycoline	V3 Glycerine	V4 Glycoline	V5 Glycoline	V6 Glycoline
Span, cm	....	1.26	1.26	1.18	1.25	1.25
Strip width, cm	....	0.0350	0.0350	0.0350	0.0372	0.0372
Strip thickness, cm	....	0.0025	0.0025	0.0025	0.0028	0.0028
Strip spacing, center to center, cm	....	0.055	0.055	0.056	0.052	0.052
Mirror length, cm	....	0.150	0.150	0.130	0.180	0.180
Mirror width, cm	....	0.065	0.065	0.080	0.060	0.060
Mirror thickness, cm	....	0.014	0.014	0.014	0.014	0.014
Strip tension, kilodynes per strip	....	51.5	51.5	74.5	39.1	96.0
Air-gap of vibrator well in cm	....	0.28	0.28	0.23	0.15	0.15
D-c. resistance, ohms	1.591	0.725	0.725	0.670	1.610	1.610
A-c. resistance at the resonant freq., ohms	1.650	0.753	0.745	0.733	1.739	1.739
f_0, ~	2280	2366	1707	2950	2170	2760
f_1, ~	1862	2010	1246	2520	1670	2240
f_2, ~	2790	2780	2340	3480	2815	3390
ω_0, rad/sec	14,320	14,840	10,720	18,520	13,630	17,320
ω_1, rad/sec	11,680	12,620	7,825	15,730	10,480	14,050
ω_2, rad/sec	17,520	17,460	14,705	21,840	17,650	21,300
Z_m, absohms	0.032×10^9	0.0219×10^9	0.0054×10^9	0.02085×10^9	0.0201×10^9	0.030×10^9
θ, rad/absamp	9.15	8.55	2.62	4.41	12.4	8.38
Δ, hyp. rad/sec	2920	2420	3440	3055	3580	3655
A, dyne-p-cm./absamp	244.5	173.0	192.5	255.0	119.0	207.0
r, dyne-p-cm./rad. per sec	1.865×10^{-3}	1.364×10^{-3}	6.85×10^{-3}	3.124×10^{-3}	0.705×10^{-3}	1.428×10^{-3}
m, gm.-cm^2	31.95×10^{-8}	28.3×10^{-8}	99.6×10^{-8}	51.15×10^{-8}	9.85×10^{-8}	19.56×10^{-8}
s, dyne-p-cm./rad	65.5	62.1	114.5	175.3	18.2	58.6
B_0	0.203	0.1295	0.320	0.1645	0.262	0.211

TABLE IV. OSCILLOGRAPHIC RESULTS

			From observations of f_0, θ_0 and θ_S						From Lissajous figures			
Vibrator	Damping fluid	Temp deg. cent.	Δ	f_0 ~	f_d ~	f_f ~	f_1 ~	f_2 ~	f_1 ~	f_2 ~	w_0	B_0
1	2	3	4	5	6	7	8	9	10	11	12	13
0—1	Castor oil	27.5	5,750	1,486	721	1,170	827	2,657	813	2,570	9,388	0.615
0—3	Air	22.5	1,010	3,235	3,220	3,228	3,079	3,401	3,100	3,420	20,320	0.050
0—4	Air	27	1,430	2,985	2,970	2,978	2,767	3,223	2,760	3,215	18,755	0.096
0—5	Air	28.5	823	2,880	2,875	2,878	2,751	3,013	2,760	3,000	18,095	0.0455
0—5	Castor oil	28	5,700	1,151	Imag	707	561	2,375	513.5	2,475	7,230	0.79
0—6	Air	25	609	4,110	4,110	4,110	4,013	4,207	4,010	4,240	25,830	0.0236
0—6	Glycoline	26	3,100	1,957	1,827	1,891	1,523	2,513	1,464	2,645	12,300	0.252
0—6	Russian Min'l Oil	28	4,400	1,614	1,272	1,454	1,060	2,460	993	....	10,140	0.434
0—6	Castor Oil	27	8,980	1,302	Imag	Imag	505	3,365	517	....	8,172	1.10
0—7	Air	26	92	2,635	2,335	2,635	2,620	2,650	2,615	2,650	16,560	0.0055
0—7	Glycoline	27	1,320	1,674	1,650	1,662	1,475	1,895	1,444	1,957	10,520	0.125
0—7	Russian Min'l Oil	25	3,000	1,614	1,460	1,540	1,202	2,158	1,246	2,133	10,140	0.288
0—7	Castor Oil	27	6,240	1,196	Imag	665	560	2,544	555	2,520	7,515	0.83
0—8	Air	27	1,620	4,210	4,195	4,203	3,962	4,468	3,830	4,570	26,460	0.0613
0—8	Russian Min'l Oil	26	4,150	2,010	1,780	1,900	1,455	2,775	1,404	2,755	12,630	0.33
0—8	Glycoline	27	3,142	2,323	2,215	2,270	1,880	2,880	1,932	2,820	14,600	0.215
0—8	Castor Oil	27	12,400	1,674	Imag	Imag	615	4,565	634	....	10,520	1.18
G	Light damping liquid	....	4,450	3,300	3,140	3,220	2,663	4,077	2,520	4,070	20,740	0.214

sistance balance to be obtained on the oscillograph vibrator O. An extraneous resistance R, connects the end of the slidewire and junction C. A zero balance is first obtained at the resonant frequency f_o, and then at any frequency well removed from resonance. The change in the setting of the slider thus produced enables the motional impedance Z absohms to be evaluated.

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SUMMARY

1. Every mechanical oscillograph vibrator is subject to a vector correction factor, at other frequencies than that at which it is calibrated, and especially at high frequencies.

2. The mechanical laws governing an oscillographic vibrator are precisely similar to those governing a simple alternating-current branch circuit of fixed C, L and R, under variable impressed frequency. The velocity-impedance graph is a straight line: The displacement-impedance graph is a parabola.

3. Every oscillogram, intended for accurate analysis, is completely defined in regard to its correction for frequency by two constants; namely, f_o the resonant frequency, and B_o the oscillatory bluntness. These two constants, or their equivalent data, should be measured, and recorded on the oscillogram.

4. If the two constants f_o and B_o,—the resonant frequency and bluntness of an oscillograph,—are measured, its behavior and correction factors for all frequencies can be computed relatively to its calibration at any particular frequency.

5. In ordinary liquid-damped oscillographs, f_o commonly lies between 1000 $\sim$ and 5000 $\sim$. The greater the damping, the lower f_o tends to become.

6. The bluntness of resonance B_o indicates the ratio of the frictional resistance present to the resistance necessary for strict aperiodicity. When $B_o = 1$, the vibrator is just aperio-

dic. The bluntness of an air-damped vibrator may be as low as 0.0014. Oscillographs damped in castor oil may readily have bluntness greater than 1, or be ultraperiodic. The best bluntness for a long-range oscillograph, in order to have relatively small corrections to be applied at moderately high frequencies is, perhaps, near $B_o = 0.6$.

7. The value of f_o for an oscillograph may conveniently be determined by means of a simple permanent-magnet auxiliary vibrator, operating from the same source, and so arranged as to supply a critical and recognizable form of Lissajous optical figure at resonance.

8. Having identified f_o for an oscillograph, the bluntness B_o may be determined, by measuring the specific deflection at a low frequency and also at the resonant frequency.

9. A number of actual oscillographs have been tested, and typical results, obtained from them, are reported.

10. In order to obtain the fundamental constants A, m, r and s, of a vibrator, it is desirable to measure its motional impedance at resonance, in addition to securing the measurements for f_o and B.

11. In analyzing the oscillograph of a periodically recurring complex harmonic wave into its Fourier components, the analysis is first made, as though the oscillograph had no error with change of frequency, and then each evaluated harmonic component should be corrected, both in magnitude and in phase position, by applying the correction factor for that frequency, in view of the values of f_o and B for the instrument.

12. A number of curves of deviation factors for different values of B have been worked out, and are presented for use, in order to supply approximate corrections by inspection.

List of Symbols Employed

A	Torque constant of a vibrator, or the torque produced by unit current $\left(\frac{\text{dyne perp. cm.}}{\text{absampere}}\angle\right)$.
$\alpha = \tan^{-1}\left(\frac{L\omega - 1/C\omega}{R}\right)$	Slope of an impedance to current (radians or degrees).
$B = \Delta/\omega_o = B_o$	Bluntness of resonance of a vibrator or branch circuit based on r_o (numeric).

$B_s = 2\,B_0$	Bluntness of resonance of a vibrator or branch circuit based on τ_s (numeric).
β	Slope of a displacement admittance, varying between $-$ 0 deg. and $-$ 180 deg. (degrees)
C	Capacitance of a condenser (farads).
$c = 1/s$	Slackness of a vibrator $\left(\dfrac{\text{radians}}{\text{dyne perp. cm.}}\right)$.
Δ	Damping constant of a vibrator, or hyp. angular velocity of decay $\left(\dfrac{\text{hyps.}}{\text{sec.}}\right)$.
$D = y'/y_s'$	Deviation factor, or ratio of vector displacement at ω to that at ω_s (numeric $\angle$)
E	Maximum cyclic e.m.f. impressed on a branch circuit (volts $\angle$).
F	Maximum cyclic torque impressed on a vibrator (dynes-perp-cm.).
f, f_0, f_1, f_2	Frequency, with reference to resonance and quadrantal values (cycles per sec.).
G	Conductance of a branch circuit (mhos).
$g = 1/r$	Mechanical conductance of a vibrator (mechanical abmhos).
I	Alternating current supplied to a branch circuit (max-cyclic amperes $\angle$). also alternating current supplied to a vibrator (max. cyclic absamperes $\angle$).
Θ	Maximum cyclic displacement produced in a vibrator mirror (radians $\angle$).
Θ_i	Initial cyclic displacement produced in a vibrator mirror (radians $\angle$).
$\dot{\Theta}$	Maximum cyclic angular velocity produced in a vibrator mirror $\left(\dfrac{\text{radians}}{\text{sec.}}\angle\right)$.
$\theta = \Theta/I$	Specific mirror deflection, or max. cy. displacement per r.m.s. absampere, produced in a vibrator $\left(\dfrac{\text{radians}}{\text{absampere}}\angle\right)$.
θ_s, θ_0	Specific deflections produced at a low frequency, and at resonant frequency $\left(\dfrac{\text{radians}}{\text{absampere}}\angle\right)$.

θ_m	Max. cy. displacement per max. cy. absampere produced in a vibrator $\left(\frac{\text{max. cy. radians}}{\text{max. cy. absamp.}}\ \angle\right)$.
$j = \sqrt{-1}$	
$K = 1/D$	Correction factor for the max. cy. displacement at any impressed ang. vel. ω, to that at ω_s (numeric ∠).
L	Constant inductance of a branch circuit (henrys).
m	Moment of inertia of a vibrator system ($\text{gm} - \text{cm.}^2$).
Q	Quantity of electricity cyclically displaced in a branch circuit (coulombs).
R	Constant resistance of a branch circuit (ohms)
r	Resistance to motion of a vibrator $\left(\text{mechanical absohms or } \frac{\text{dyne-perp.-cm.}}{\text{radian per sec.}}\right)$
s	Stiffness coefficient of a vibrator $\left(\frac{\text{dynes perp. cm.}}{\text{radian}}\right)$.
t	Elapsed time (seconds).
$\tau = \tau_0 = 1/\Delta$ $= \frac{2L}{R}$	Oscillatory time constant of a circuit or system $\left(\frac{\text{seconds}}{\text{hyp. radian}} \text{ or seconds}\right)$.
$\tau_s = \tau_0/2 = L/R$	Nonoscillatory time constant of a circuit (seconds).
$u = \omega/\omega_0$	Ratio of an impressed angular velocity to the resonant angular velocity of a system (numeric).
$\Lambda = \omega_0/\Delta$ $= \Lambda_0 = 2\Lambda_s$	Sharpness of resonance of a branch circuit or vibrator based on τ_0 (numeric).
$\Lambda_s = \Lambda_0/2$	Sharpness of resonance of a branch circuit or vibrator based on τ_s (numeric).

v.m.t.	Vibromotive torque (dynes perp. cm.).
$Y = 1/Z$	Admittance to alternating current in a branch circuit (mhos $\angle$).
$y = 1/z$	Admittance to angular velocity in a vibrator $\left(\text{mechanical abmhos or } \frac{\text{radians per sec.}}{\text{dynes perp. cm.}} \angle\right)$
$Y' = 1/Z'$	Admittance to displacement in a branch circircuit $\left(\frac{\text{coulombs}}{\text{volts}} \angle\right)$.
$y' = 1/z'$	Admittance to displacement in a vibrator $\left(\frac{\text{mirror radians}}{\text{dyne perp. cm.}} \angle\right)$.
Z	Impedance to alternating current in a branch circuit (ohms $\angle$).
z	Impedance to angular velocity in a vibrator $\left(\text{mechanical absohms or } \frac{\text{dynes-perp. cm.}}{\text{radians per sec.}} \angle\right)$.
Z'	Impedance to displacement in a branch circuit $\left(\frac{\text{volts}}{\text{coulombs}} \angle\right)$.
z'	Impedance to displacement in a branch vibrator $\left(\frac{\text{dynes perp. cm.}}{\text{radian}} \angle\right)$.
ω	Impressed angular velocity of circuit or vibratory system $\left(\frac{\text{radians}}{\text{sec.}}\right)$.
$\omega_0,\ \omega_1,\ \omega_2$	Impressed angular velocity of resonance and quadrantal values $\left(\frac{\text{radians}}{\text{sec.}}\right)$.
$\angle$	Sign of planevector or complex quantities.

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